Math 249 Lecture 27 Notes

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1 The Cycle Index

1.1 The cycle index and $\mathbb{C}X$

Let $S_n \circlearrowright X$, where X is finite. We can extend this action linearly to make $S_n \circlearrowright \mathbb{C}X$. This has a character $\chi_{\mathbb{C}X}$. If F is the Frobenius characteristic map, then

$$F(\chi_{\mathbb{C}X}) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_{\mathbb{C}X}(\sigma) p_{\gamma(\sigma)} = \frac{1}{n!} \sum_{\sigma \in S_n} |X^\sigma| \, p_{\gamma(\sigma)},$$

where $|X^\sigma|$ is the number of elements of X fixed by σ.

Definition 1.1. Given a species F, define the *cycle index* to be

$$Z_F(p_1, p_2, p_3, \dots) := \sum_n \frac{1}{n!} \sum_{\sigma \in S_n} |F([n])^\sigma| \, p_{\gamma(\sigma)}.$$

As a special case, the exponential generating function of F is

$$F(x) = \sum_n |F([n])| \frac{x^n}{n!} = Z_F(x, 0, 0, \dots).$$

Consider the number of unlabeled F structures on S, where $|S| = n$. What this really means is the number of S_n orbits on $F([n])$. The orbit sums in $X = F([n])$ are a basis of $(\mathbb{C}X)^{S_n}$; i.e. $\dim((\mathbb{C}X)^{S_n})$ is the number of S_n orbits on X. This is actually a general representation-theoretic property.

Proposition 1.1. *The number of G-orbits on X is equal to* $\dim((\mathbb{C}X)^G)$.

Proof. Since $\langle \chi_V, \chi_W \rangle = \dim \operatorname{Hom}_G(V, W)$, we have that

$$\dim((\mathbb{C}X)^G) = \langle \mathbb{1}_G, \chi_{\mathbb{C}X} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{\mathbb{C}X}(g) \mathbb{1}(g^{-1}) = \frac{1}{|G|} \sum_{g \in G} |X^g| = \{G\text{-orbits on } X\},$$

where the last step uses Burnside's lemma. $\square$

If we define

$$U(x) = \sum_n |F([n])/S_n|\, x^n,$$

then we have

$$U(x) = Z_F(x, x^2, x^3, \dots).$$

1.2 Examples of cycle indices

Example 1.1. Let X_k be the indicator species. Then, by what we proved earlier about the Frobenius characteristic map, $Z_{X_k} = h_k$. In particular, $Z_{X_1} = h_1 = p_1$.

Example 1.2. Let E be the trivial species. Then

$$Z_E = 1 + h_1 + h_2 + \cdots = \Omega = \prod_i \frac{1}{1 - x_i}.$$

Example 1.3. Let L be the species of linear orderings. Then

$$Z_L = \frac{1}{1 - p_1}.$$

The fact that this is a function of p_1 only is reflective of the fact that there are no automorphisms. The ordinary generating function and the exponential generating function for linear orderings are the same because since there are no automorphisms, there are $n!$ labelings for each ordering.

Example 1.4. Let P be the species of permutations. The action of S_n here is conjugation. So we are looking for the number of elements fixed by conjugation. So

$$Z_P = \frac{1}{n!} \sum_{\sigma \in S_n} |\{\tau : \sigma\tau = \sigma\}|\, p_{\gamma(\sigma)},$$

where the coeficient of p_λ is the number of pairs (σ, τ) where $\sigma \in C_\lambda$ and τ commutes with σ. This is $|C_\lambda|\,|\mathrm{Stab}(\sigma_\lambda)| = n!$ for all λ by the orbit-stabilizer theorem. So

$$Z_P = \sum_\lambda p_\lambda = \prod_k \frac{1}{1 - p_k}.$$

This means that

$$U(x) = \sum_n p(n) x^n = \prod_k \frac{1}{1 - x^k},$$

where $p(n)$ is the number of partitions of n.

Proposition 1.2. *Let A, B be species. Then*

$$Z_{A+B} = Z_A + Z_B,$$

$$Z_{AB} = Z_A Z_B.$$

We will prove these later. For now, let's see a consequence.

Example 1.5. , t Recall that if L is the species of linear orderings, then $L \cong X_0 + X_1 L$, so $L(x) = 1 + xL(x)$, which we can solve to get $L(x) = \frac{1}{1-x}$. Similarlyhis species isomorphism also gives that

$$Z_L = Z_{X_0} + Z_X Z_L = 1 + p_1 Z_L,$$

so we can solve to get

$$Z_L = \frac{1}{1 - p_1}$$

in a different way from before.

1.3 Plethystic substitution

This is sometimes also called λ-ring substitution.

Let A be an expression with variables. A could be a polynomial, an element of the field of rational functions, a formal Laurent series, etc. We require the ring containing A to have some well-defined notion of a homomorphism sending variables to their k-th power.

Then let $p_k[A]$ be A with variables $a_1 \mapsto a_i^k$, and given f, let

$$f[A] = f(p_1[A], p_2[A], \cdots).$$

Then $f \in \Lambda_{\mathbb{Z}} = \mathbb{Z}[p_1, p_2, \dots]$.

Example 1.6. Let $X = x_1 + x_2 + \cdots + x_n$. Then

$$f[X] = f|_{p_k \mapsto p_k(x_1,\dots,x_n)} = f(x_1, \dots, x_n).$$

Recall that $\omega p_k = (-1)^{k-1} p_k$. So

$$\begin{aligned} f[-X] &= f|_{p_k \mapsto -p_k(x_1,\dots,x_n)} = f|_{p_k \mapsto \omega p_k(-x_1,\dots,-x_n)} = (\omega f)(-x_1, \dots, -x_n)) \\ &= (\omega f)[X]|_{x_i \mapsto -x_i}. \end{aligned}$$

In fact, $f[-tX] = \omega f[tX]|_{t \mapsto -z}$.